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Quantum mechanics for genus $g = 2$-persistent current in coupled rings

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Abstract

We solve the quantum mechanical problem for electrons confined to two coupled metallic rings which form an '8' structure. The solution is achieved by gluing the wavefunction at the interface of the two rings using a modification of Dirac's constrained method. The Heisenberg equation of motion for the wavefunction is a fermionic and bosonic mixture, suggesting that it is impossible to solve the problem for two coupled rings by standard boundary conditions.

As an explicit demonstration of our method we present an exact solution for the persistent current on coupled rings in the presence of two external magnetic fluxes. For large coupled rings with equal fluxes we find that the persistent current in the two coupled rings is equal to that in a single ring. For opposite fluxes the energy has a chaotic structure. For both cases the periodicity is h/e.

In order to compare our theory to the experimental situations we consider two rings with a finite width in the ballistic regime in the presence of $2K_F$ impurity scattering.

1. Introduction

The physics of a newly fabricated material such as graphene [1] and nanotubes [2] requires exact methods for computing the wavefunction which is essential to determine their physical properties. In particular the wavefunction coherence at low temperatures in the ballistic regime is sensitive to the geometrical topology and is path-dependent. One of the important topological characterizations is the number of holes on a closed surface. The number of holes formed thereon is often referred to as a genus number g. In the past most of the studies have been devoted to the $g = 1$ case. Examples of $g = 1$ are given by a Torus geometry and $g = 2$ corresponds to a double Torus used for the two-dimensional quantum Hall problem [3].

In one dimension the $g = 1$ corresponds to the Aharonov–Bohm geometry [4] leading to a famous theorem [5] that in the presence of an external flux all the physical properties are periodic with the flux period $\Phi_0 = h/e$ independent from microscopic details such as electron–electron interactions or disorder. Recently the $g = 1$ case has been investigated for a ring structure of graphene [6] while genus $g = 2$ has been realized in carbon nanotubes [7].

We believe that the method for investigating high genus material should be based on gluing the wavefunction between different sectors. This can be achieved by using the method of constraints. Constraints have a long history in condensed matter physics. In particular it has been shown that, using constraints, strong coupling problems such as the Hubbard model can be solved [8–13].

Here we propose to investigate the wavefunction for genus $g = 2$ with the simplest geometry being two rings perfectly glued at one point to form a character '8' structure. Our main result for this case is that, independent of the form of the Hamiltonian, the solution for the wavefunction is given in terms of the modified Heisenberg equation of motion. For any fermionic operator $\widehat{O}_F$ in the presence of two fermionic second-class constraints Q^+ and Q which obey $[Q, Q^+]_+ = QQ^+ + Q^+Q \neq 0$ and a Hamiltonian H we find $i\hbar\frac{d\widehat{O}_F}{dt} = [\widehat{O}_F, H]_D$.

The symbol $[,]_D$ stands for a mixture of commutators $[,]$ and anti-commutators $[,]_+$. The explicit form of the new commutator is given by

$$\left[\widehat{O}_F, H\right]_D \equiv [\widehat{O}_F, H] - [\widehat{O}_F, Q]_+([Q, Q^+]_+)^{-1}[Q^+, H] - [\widehat{O}_F, Q^+]_+([Q^+, Q]_+)^{-1}[Q, H].$$

This result suggests that the gluing of the wavefunction cannot be described within the regular method of matching boundary conditions! To understand why this is the case we assume that the gluing of the wavefunction is introduced with the help of delta function potentials which enforce the constraints. The potentials acts as Lagrange multipliers. For a

fixed Lagrange multiplier the problem reduces to a quantum mechanical problem with a delta function potential. Such a problem can be solved using boundary conditions. But a physical result is obtained only after we average over all the possible delta function potentials.

Our solution is obtained using a modification of the Dirac bracket [18] for electronic systems. Enforcing the constraint by a delta function allows the identification of an effective physical model, of a capacitor connecting the two rings. In the limit of a zero capacitance an infinite impurity Hubbard [8–11] interaction (at the contact point) is obtained once we perform the formal identification of the ring index with the spin index.

As an explicit demonstration for the $g = 2$ structure we choose to solve the problem of persistent currents. The $g = 1$ case in one dimension is represented by the Aharonov–Bohm geometry which in the presence of an external magnetic flux gives rise to a non-dissipative current for mesoscopic metallic rings [12–17, 21] named the persistent current. To the best of our knowledge the persistent current problem for a genus $g = 2$ has not been considered before. On the other hand a variety of geometries of coupled rings have been considered in the literature [15, 16, 21]. In particular we mention the example of two coupled rings which are connected by two arms which have a common grounded point [15]. Due to this grounded point the genus $g = 2$ topology is lost. For this case it was possible to use current loop equations and solve the current problem using boundary conditions. In our case no grounded point exist and therefore no simple boundary conditions can be used.

In this paper, we report the exact solution for multiple connected geometries, such as a geometry with two holes two rings perfectly glued at one point to form a character '8' structure. The geometry modifies the global properties of the wavefunction, and the presence of magnetic fluxes generates persistent currents with complicated periods. We present an exact analytical solution for the eigenvalues and compute the persistent current for two coupled rings with a character '8' structure for two different fluxes. We solve the problem by modeling the gluing of the two rings using fermionic constraints with anti-commuting Lagrange multipliers which can be viewed as a capacitive impurity model $E_C = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$ coupled to the two rings in the limit of a zero capacitance C_0 where $\rho_1(x = 0)$ is the density at the point contact for ring one and $\rho_2(x = 0)$ is the density for the second ring.

The analytical results are investigated numerically using Mathematica. When the two fluxes on both rings are the same, we find a simple relation between the single-ring ($g = 1$) current $I^{(g=1)}(\text{flux}; N)$ and the double-ring ($g = 2$) current $I^{(g=2)}(\text{flux}; N)$. At $T = 0$ we define $I^{(g=2)}(\text{flux}; N) = r(N)I^{(g=1)}(\text{flux}; N)$ where $r(N)$ is the ratio between the two currents. The ratio $r(N)$ is a function of the number of sites N and obeys $r(N) \to 1$ for $N \to \infty$. The effects of the electron–electron interaction away from half-filling can be investigated within a Luttinger liquid theory. The point contact between the rings is described by a delta function potential where the strength of the potential is given by a Lagrange multiplier. This situation is equivalent to an effective Gaussian impurity in a Luttinger liquid theory. Contrary to a regular impurity the Gaussian point contact impurity is an irrelevant perturbation for moderate interactions, (only for interactions which have an interaction parameter $K_c < 1/2$ can it suppress the current). This means that for large rings and moderate electron–electron interactions $1/2 < K_c < 1$ the effect of the point contact will not affect the current amplitudes. This situation changes when we allow for real impurity scatterings. Using the Luttinger liquid theory presented in [19] we showed that the presence of a $2K_F$ impurity scattering in a Luttinger liquid gives rise to a drastic enhancement of the kinetic mass of the zero mode [19], resulting in a drastic decrease of the current amplitudes in each ring.

We have used the results given in [19] to compare our theory to the experimental situations. We have shown that in the ballistic regime the effects of the $2K_F$ impurity scattering in a multichannel Luttinger theory (due to the finite width of the rings) gives a good agreement with the current observed in the experiment of [17]. This allows us to propose a ballistic (non-diffusive) explanation for the experiment of a line of GaAs/GaAlAs coupled rings [17].

The plan of this paper is as followings: in section 2 we present the exact analytical results for the two rings perfectly glued at one point to form a character '8' structure.

In section 3 we present the application of our theory to the problem of persistent currents for two coupled rings in the presence of two external fluxes.

In section 4 we present our numerical results for two coupled rings for equal and opposite fluxes.

Section 5 is devoted to discussions.

We show that the method presented can be extended to two rings with a finite width.

Using the results given in [19] (for a one-dimensional persistent current with a $2K_F$ impurity scattering) we extend the results to a multichannel persistent current problem.

We show that the theory developed for the ballistic regime might explain the results for the GaAs/GaAlAs coupled rings [17].

A generalization of our method to genus $g > 2$ multicoupled rings is presented.

The constraint for the two rings can be understood as a capacitor model $E_C = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$ ($\rho_\alpha(x = 0), \alpha = 1, 2$ is the charge on each ring). This problem is analyzed by the method of matching the boundary conditions. A physically universal result is obtained only after the limit $C_0 \to 0$ is taken or the average over all the possible potentials is performed.

2. Exact solution for two rings perfectly glued to form a character '8' structure

In this section we present the solution of two coupled rings for an arbitrary Hamiltonian. We consider first a single channel and show that the results are applicable for rings with a finite width.

Dirac has shown that for the second-class constraints the Poisson brackets are replaced by the Dirac brackets [18]. For an even number of constraints q_r with non-zero Poisson brackets one finds that the equations of motions are governed

by the Dirac [18] brackets which replace the Poisson bracket $\{A, B\}$ by $\{A, B\}_D = \{A, B\} - \sum_{r,r'}\{A, q_r\}c_{rr'}\{q_{r'}, B\}$. The matrix $c_{rr'}$ is the inverse of the Poisson bracket $\{q_r, q_s\}$ and is computed from the equation $\sum_{r'} c_{rr'}\{q_{r'}, q_s\} = \delta_{r,s}$. A quantum theory is obtained by replacing the Poisson bracket by the commutators $\{,\} = \mathrm{i}\hbar[,]$.

We will present results for electronic systems with second-class constraints. Our calculations allow us to propose that for the second-class fermionic constraints a modified Heisenberg equation of motion should be used. Given two fermionic constraints Q, Q^+ which obey non-zero anti-commutation relations $[Q, Q^+]_+ \equiv QQ^+ + Q^+Q \neq 0$ we find that the Dirac bracket is modified. The new Heisenberg equation of motion for a fermionic operator $\widehat{O}_{\mathrm{F}}$ and a Hamiltonian H is given by

$$\mathrm{i}\hbar\frac{\mathrm{d}\widehat{O}_{\mathrm{F}}}{\mathrm{d}t} = \left[\widehat{O}_{\mathrm{F}}, H\right]_D$$

where

$$\left[\widehat{O}_{\mathrm{F}}, H\right]_D = \left[\widehat{O}_{\mathrm{F}}, H\right] - \left[\widehat{O}_{\mathrm{F}}, Q\right]_+ \left(\left[Q, Q^+\right]_+\right)^{-1}\left[Q^+, H\right] - \left[\widehat{O}_{\mathrm{F}}, Q^+\right]_+ \left(\left[Q^+, Q\right]_+\right)^{-1}\left[Q, H\right].$$

In the first part we present the derivation of this new result.

Each ring obeys periodic boundary conditions. For each ring, the point x is identified with the point $x + L$. The two coupled rings with the character '8' structure (i.e. $g = 2$) are obtained by identifying the middle point $x = L/2$ of the first ring with point $x = 0$ of the second ring, i.e. $C_1(L/2) = C_2(0)$ and $C_1^+(L/2) = C_2^+(0)$. The operators $C_1(L/2)$ and $C_1^+(L/2)$ represent the electronic annihilation and creation operators for the first ring at the point $x = L/2$. For the second ring at the point $x = 0$ we introduce the electronic operators $C_2(0)$ and $C_2^+(0)$. This identification is equivalent to two fermionic constraints, $Q \equiv C_1(L/2) - C_2(0)$ and $Q^+ \equiv C_1^+(L/2) - C_2^+(0)$ which are enforced by anti-commuting Lagrange multipliers, μ^+ and μ. Following [18] we replace the Hamiltonian H for the two uncoupled rings by a new Hamiltonian H_T given by

$$H_T = H + \mu^+ Q + Q^+\mu.$$

The constraint conditions are implemented with the help of the anti-commuting Lagrange multipliers μ and μ^+. The unusual physical meaning of the anti-commuting Lagrange multipliers can be viewed as a fermionic impurity [18] which mediates the hopping of the electrons between the two rings.

The wavefunction for the genus $g = 2$ problem is given by the eigenstate $|\chi\rangle$ of the Hamiltonian for the two rings with the additional conditions

$$Q|\chi\rangle = 0 \qquad \text{and} \qquad Q^+|\chi\rangle = 0.$$

The Hamiltonian H_T is used to determine the Lagrange multipliers which are determined by the condition that the constraints must be satisfied at any time. Therefore the time derivative of constraints must be zero at any time

$$\dot{Q}|\chi\rangle = \dot{Q}^+|\chi\rangle = 0.$$

We introduce the notations $[A, B]_+ \equiv AB + BA$ and $[A, B] = AB - BA$.

The Heisenberg equation of motion for the constraint Q is

$$\begin{aligned}\mathrm{i}\hbar\dot{Q} &= [Q, H_T] = [Q, H] + [Q, \mu^+Q + Q^+\mu] \\ &= [Q, H] + [Q, \mu^+]_+Q - \mu^+[Q, Q]_+ + [Q, Q^+]_+\mu \\ &\quad - Q^+[Q, \mu]_+ = [Q, H] + [Q, Q^+]_+\mu. \end{aligned} \tag{1}$$

The anti-commuting Lagrange multipliers satisfy the equations; $[Q, \mu^+]_+ = [Q, \mu]_+ = [Q^+, \mu^+]_+ = [Q^+, \mu]_+ = 0$. The constraints are fermionic and obey

$$[Q, Q^+]_+ = [Q^+, Q]_+ = 2.$$

This result shows that the constraints are second-class constraints [18]. From the condition $\dot{Q}|\chi\rangle = 0$ and equation (1) we determine the Lagrange multiplier field μ:

$$\mu = -[Q^+, Q]_+^{-1}[Q, H] = -\tfrac{1}{2}[Q, H].$$

The field μ^+ is obtained from the equation $\dot{Q}^+|\chi\rangle = 0$:

$$\mu^+ = [Q, Q^+]_+^{-1}[Q^+, H] = \tfrac{1}{2}[Q^+, H].$$

The Hamiltonian H_T with the constraints and the Lagrange multipliers are used to compute the Heisenberg equation of motion for any fermionic operator $\hat{O}$ (the Lagrange multipliers anti-commute with any fermionic operator, i.e. $[\hat{O}, \mu]_+ = [\hat{O}, \mu^+]_+ = 0$):

$$\begin{aligned}\mathrm{i}\hbar\frac{\mathrm{d}\hat{O}}{\mathrm{d}t} &= [\hat{O}, H_T] = [\hat{O}, H] + [\hat{O}, \mu^+Q] + [\hat{O}, Q^+]\mu \\ &= [\hat{O}, H] + [\hat{O}, \mu^+]_+Q - \mu^+[\hat{O}, Q]_+ \\ &\quad + [\hat{O}, Q^+]_+\mu - Q^+[\hat{O}, \mu]_+ \\ &= [\hat{O}, H] - [\hat{O}, Q]_+\mu^+ - [\hat{O}, Q^+]\mu. \end{aligned} \tag{2}$$

We substitute in equation (2) the solutions for the Lagrange multiplier fields and obtain a new equation of motion with a new commutator which resemble the classical Dirac brackets [18]:

$$\begin{aligned}\mathrm{i}\hbar\frac{\mathrm{d}\hat{O}}{\mathrm{d}t} &= [\hat{O}, H_T] = [\hat{O}, H] - [\hat{O}, Q^+]_+([Q^+, Q]_+)^{-1} \\ &\quad \times [Q, H] - [\hat{O}, Q]_+([Q, Q^+]_+)^{-1}[Q^+, H] \\ &\equiv [\hat{O}, H]_D. \end{aligned} \tag{3}$$

Equation (3) shows that the Heisenberg equation of motion for the fermionic operator is governed by a new commutator $[\hat{O}, H]_D$. The equations $Q|\chi\rangle = 0$ and $Q^+|\chi\rangle = 0$ are inconsistent with $[Q, Q^+]_+|\chi\rangle \neq 0$ since the anti-commutator of the constraints is non-zero. The new anti-commutator $[Q, Q^+]_{+,D}$ resolves the inconsistency problem

$$\begin{aligned}[Q, Q^+]_{+,D}|\chi\rangle &\equiv [Q, Q^+]_+|\chi\rangle - [Q, Q^+]_+([Q^+, Q]_+)^{-1} \\ &\quad \times [Q, Q^+]_+|\chi\rangle - [Q, Q]_+([Q, Q^+]_+)^{-1} \\ &\quad \times [Q^+, Q^+]_+|\chi\rangle = 0. \end{aligned}$$

This result can be generalized for any operator X which is either bosonic or fermionic in the presence of any two second-class constraints Q and Q^+.

We introduce a compact notation for the commutator or the anti-commutator:

$$[A,B]_{\epsilon(AB)} \equiv AB - (-1)^{n(A)n(B)}BA$$

where $n(A)=0$ and $n(B)=0$ for bosons

and $n(A)=1$ and $n(B)=1$ for fermions.

In this new notation the constraint field commutator or anti-commutator obey $[Q,Q^+]_{\epsilon(QQ^+)} \neq 0$.

As a result the Heisenberg equation of motion for the operator X will be replaced by

$$i\hbar\frac{dX}{dt} = [X,H_T]_{\epsilon(D)}$$

where the generalized Dirac commutator takes the form

$$[X,H_T]_{\epsilon(D)} \equiv [X,H] - [X,Q^+]_{\epsilon(XQ^+)}([Q^+,Q]_{\epsilon(Q^+Q)})^{-1} \times [Q,H] - [X,Q]_{\epsilon(XQ)}([Q,Q^+]_{\epsilon(QQ^+)})^{-1}[Q^+,H].$$

For the remaining part we will confine ourself to the fermionic case given in equation (3). We will compute the Heisenberg equations of motion for the creation and annihilation fermionic operators $C_\alpha(x,t)$ and $C^+_\alpha(x,t)$ $\alpha = 1,2$ which obey periodic boundary conditions $C_\alpha(x) = C_\alpha(x+L)$ and $C^+_\alpha(x) = C^+_\alpha(x+L)$ and $\alpha = 1,2$ is the index of the rings. For H we can use any Hamiltonian for the two uncoupled rings:

$$i\hbar\dot{C}_\alpha(x) = [C_\alpha(x),H]_D = [C_\alpha(x),H] - \tfrac{1}{2}[C_\alpha(x),Q^+]_+[Q,H].$$

The effects of the finite width of the rings is taken into consideration in the following way. For this situation we will introduce the y coordinate in the transversal direction. The constraints Q and Q^+ are given in terms of the single particle operator $C_\alpha(x,y)$, $C^+_\alpha(x,y)$ which is a function of the coordinate x along the ring and y is the coordinate in the transversal direction. The modified constraints are a function of the coordinate y:

$$Q(y) = C_{\alpha=1}(x=0,y) - C_{\alpha=2}(x=L/2,y) \quad \text{and}$$
$$Q^+(y) = C^+_{\alpha=1}(x=0,y) - C^+_{\alpha=2}(x=L/2,y).$$

3. An application to persistent currents—exact solution for two rings in the presence of an external magnetic flux

We consider two rings of length L which are threaded by a magnetic flux Φ_α, where $\alpha = 1,2$ (for each ring). In order to observe the changes of the constraints in the presence of the external flux, we perform the following steps. In the absence of the external flux $\Phi_\alpha = 1,2$ the Hamiltonian for the two rings is given by $H_0 = -t\sum_{\alpha=1}^{2}\sum_{x=0}^{(N_s-1)a}[C^+_\alpha(x)C_\alpha(x+a) + \text{h.c.}]$. The length of each ring is $L = N_s a$, where N_s is the number of sites and a is the lattice spacing. When the external magnetic flux Φ_α is applied the Hamiltonian H_0 is replaced by H. The Hamiltonian H is obtained by the transformation $C_\alpha(x) \to \exp[i\frac{e}{\hbar c}\int_0^x A(x';\alpha)\,dx']C_\alpha(x) \equiv \psi_\alpha(x)$ and $C^+_\alpha(x) \to C^+_\alpha(x)\exp[-i\frac{e}{\hbar c}\int_0^x A(x';\alpha)\,dx'] \equiv \psi^+_\alpha(x)$. Here $A(x;\alpha)$ is the tangential component of the vector potential on each ring. The relation between the flux and the vector potential on each ring is $\frac{e}{\hbar c}\int_0^L A(x;\alpha)\,dx = \varphi_\alpha$.

The flux Φ_α on each ring $\alpha = 1,2$ gives rise to a change in the boundary conditions, $\psi_\alpha(x+N_s a) = \psi_\alpha(x)e^{i\varphi_\alpha}$ and $\psi^+_\alpha(x+N_s a) = \psi^+_\alpha(x)e^{-i\varphi_\alpha}$, where $\varphi_\alpha = 2\pi(\frac{e\Phi_\alpha}{\hbar c}) = 2\pi\frac{\Phi_\alpha}{\Phi_0} \equiv 2\pi\hat{\varphi}_\alpha$. This boundary condition gives rise to a normal mode expression for each ring, $\psi_\alpha(x) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N_s-1} e^{iK(n,\varphi_\alpha)\cdot x}\psi_\alpha(n)$ and a similar expression for $\psi^+_\alpha(x)$. The 'momentum' $K(n,\varphi_\alpha)$ is given by $K(n,\varphi_\alpha) = \frac{2\pi}{N_s a}(n+\hat{\varphi}_\alpha)$ where $n = 0,1,\ldots,N-1$ are integers with $N = N_s$ and $\varphi_\alpha = 2\pi\hat{\varphi}_\alpha$. In the momentum space the fermionic operators $\psi_\alpha(n)$ and $\psi^+_\beta(m)$ obey anti-commutation relations $[\psi_\alpha(n),\psi^+_\beta(m)]_+ = \delta_{\alpha,\beta}\delta_{n,m}$. The Hamiltonian for the two rings in the presence of the external flux takes the form

$$H = -t\sum_{\alpha=1,2}\sum_{x=0}^{(N_s-1)a}\psi^+_\alpha(x)\psi_\alpha(x+a) + \text{h.c.}$$
$$= \sum_{\alpha=1,2}\sum_{n=0}^{N_s-1}\epsilon(n,\hat{\varphi}_\alpha)\psi^+_\alpha(n)\psi_\alpha(n) \tag{4}$$

where $\epsilon(n,\varphi_\alpha) = -2t\cos[\frac{2\pi}{N}(n+\hat{\varphi}_\alpha)]$ are the eigenvalues for each ring. The Hamiltonian in equation (4) has to be solved together with the transformed constraints $Q = \psi_1(\frac{L}{2})e^{-i\frac{e}{\hbar c}\int_0^{\frac{L}{2}}A(x;\alpha=1)dx} - \psi_2(0)$ and $Q^+ = \psi^+_1(\frac{L}{2})e^{i\frac{e}{\hbar c}\int_0^{\frac{L}{2}}A(x;\alpha=1)dx} - \psi^+_2(0)$.

The wavefunction for the genus $g=2$ problem is given by the eigenstate $|\chi\rangle$ of the Hamiltonian in equation (4), which in addition satisfies the equations $Q|\chi\rangle = 0$ and $Q^+|\chi\rangle = 0$. The analysis given in section 2 shows that the equation of motion is governed by equation (3). We find for the creation $\psi^+_\alpha(x,t)$ and annihilation $\psi_\alpha(x,t)$ operators the following Heisenberg equations of motion:

$$\begin{aligned} i\hbar\dot{\psi}_\alpha(x) &= [\psi_\alpha(x),H]_D = [\psi_\alpha(x),H] - \tfrac{1}{2}[\psi_\alpha(x),Q^+]_+[Q,H] \\ &= -t[\psi_\alpha(x+a)+\psi_\alpha(x-a)] - \tfrac{1}{2}[\delta_{\alpha,1}\delta_{x,L/2}e^{i\varphi_1} - \delta_{\alpha,2}\delta_{x,0}] \\ &\times(-t)\left\{e^{-i\varphi_1}\left[\psi_1\left(\frac{L}{2}+a\right)+\psi_1\left(\frac{L}{2}-a\right)\right] + e^{-i\varphi_2}\left[\psi_2\left(\frac{L}{2}+a\right)+\psi_2\left(\frac{L}{2}-a\right)\right]\right\}. \end{aligned} \tag{5}$$

The ground state wavefunction is obtained from the one-electron state $|\chi\rangle = \sum_{\alpha=1,2}\sum_{x=0}^{(N_s-1)a}Z_\alpha(x)\psi^+_\alpha(x)|0\rangle$ given in terms of the site amplitudes $Z_\alpha(x)$. Using a normal mode momentum expansion $f_\alpha(n)$ i.e. $Z_\alpha(x) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}e^{iK(n,\hat{\varphi}_\alpha)x}f_\alpha(n)$ we find the following equations for the eigenvalues λ and the amplitudes in the momentum space $f_\alpha(n)$:

$$(\lambda - \epsilon(\ell+\hat{\varphi}_1))f_1(\ell) = -\frac{e^{i\pi\ell}}{2N}\sum_{n=0}^{N-1}\epsilon(n+\hat{\varphi}_1)e^{i\pi n}f_1(n) - \frac{1}{2N}\sum_{n=0}^{N-1}\epsilon(n+\hat{\varphi}_2)f_2(n) \tag{6}$$

and

$$(\lambda - \epsilon(\ell + \hat{\varphi}_2))\, f_2(\ell) = \frac{1}{2N}\sum_{n=0}^{N-1} \epsilon(n + \hat{\varphi}_2)\, f_2(n) + \frac{e^{i\pi\ell}}{2N}\sum_{n=0}^{N-1} \epsilon(n + \hat{\varphi}_1) e^{i\pi n} f_1(n). \quad (7)$$

We diagonalize equations (6) and (7) by the linear transformations $S_1(\hat{\varphi}_1, \lambda) = -\sum_{\ell=0}^{N-1} \epsilon(\ell + \hat{\varphi}_1) e^{i\pi\ell} f_1(\ell)$ and $S_2(\hat{\varphi}_2, \lambda) = -\sum_{\ell=0}^{N-1} \epsilon(\ell + \hat{\varphi}_2) f_2(\ell)$. As a result we obtain the eigenvalue equation $\mathbf{M}\binom{S_1}{S_2} = 0$

where the matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} -(1 + \Delta_1^{(+)}) & \Delta_1^{(-)} \\ \Delta_2^{(-)} & 1 - \Delta_2^{(+)} \end{pmatrix}.$$

Here we define $\Delta_\alpha^{(+)}(\hat{\varphi}_\alpha, \lambda) \equiv \Delta_\alpha^{(\mathrm{even})}(\hat{\varphi}_\alpha, \lambda) + \Delta_\alpha^{(\mathrm{odd})}(\hat{\varphi}_\alpha, \lambda)$ and $\Delta_\alpha^{(-)}(\hat{\varphi}_\alpha, \lambda) \equiv \Delta_\alpha^{(\mathrm{even})}(\hat{\varphi}_\alpha, \lambda) - \Delta_\alpha^{(\mathrm{odd})}(\hat{\varphi}_\alpha, \lambda)$ with the even and odd representations given by $\Delta_\alpha^{(\mathrm{even})}(\hat{\varphi}_\alpha, \lambda) = \frac{1}{2N}\sum_{m=0}^{(N-2)/2} \frac{\epsilon(2m+\hat{\varphi}_\alpha)}{\lambda - \epsilon(2m+\hat{\varphi}_\alpha)}$ and $\Delta_\alpha^{(\mathrm{odd})}(\hat{\varphi}_\alpha, \lambda) = \frac{1}{2N}\sum_{m=0}^{(N-2)/2} \frac{\epsilon(2m+1+\hat{\varphi}_\alpha)}{\lambda - \epsilon(2m+1+\hat{\varphi}_\alpha)}$. We compute $\det \mathbf{M} = 0$ and obtain the characteristic polynomial which is used to determine the eigenvalues λ:

$$2\left[\Delta_1^{(\mathrm{even})}(\hat{\varphi}_1, \lambda)\Delta_2^{(\mathrm{odd})}(\hat{\varphi}_2, \lambda) + \Delta_1^{(\mathrm{odd})}(\hat{\varphi}_1, \lambda)\Delta_2^{(\mathrm{even})}(\hat{\varphi}_2, \lambda)\right] + \left[\Delta_1^{(+)}(\hat{\varphi}_1, \lambda) - \Delta_2^{(+)}(\hat{\varphi}_2, \lambda)\right] = 1. \quad (8)$$

Equation (8) is our main result for the genus $g = 2$ case. The matrix M is *Hermitian* and the eigenvalues are real. For the particular case where the fluxes are equal, i.e. $\hat{\varphi}_1 = \hat{\varphi}_2$, or opposite, i.e. $\hat{\varphi}_1 = -\hat{\varphi}_2$, it is easy to see that the matrix M is *symmetric*.

4. Numerical analysis

We have numerically solved the secular equation (8) where N_s represents the number of sites and a is the lattice constant. To compute the current, we sum over the current carried by each eigenvalue $\lambda(\hat{\varphi}_1, \hat{\varphi}_2)$ using the grand-canonical ensemble with a fixed chemical potential. In the absence of disorder the difference between a canonical ensemble calculation and a grand-canonical one is insignificant. The current in each ring $\alpha = 1, 2$ is given by $I_\alpha^{(g=2)}(\hat{\varphi}_1, \hat{\varphi}_2) = -\sum_{\lambda(\hat{\varphi}_1,\hat{\varphi}_2)} \frac{\mathrm{d}}{\mathrm{d}\hat{\varphi}_\alpha}[\lambda(\hat{\varphi}_1, \hat{\varphi}_2)] F(\frac{(\lambda(\hat{\varphi}_1,\hat{\varphi}_2) - E_F)}{K_B T})$ where $F(\frac{(\lambda(\hat{\varphi}_1,\hat{\varphi}_2) - E_F)}{K_B T})$ is the Fermi Dirac function with the chemical potential E_F and temperature T. The current is sensitive to the number of electrons being either even or odd. We use the grand-canonical ensemble and limit ourselves to a situation with even numbers of sites and a zero chemical potential, i.e. $E_F = 0$ (which corresponds to the half-filled case). In order to have a perfect particle–hole symmetry, we will restrict the analysis to the special series for the number of sites being $N_s = 2, 6, 10, 14, 18, \ldots, 2m + 2$, where $m = 0, 1, 2, 3 \ldots$. For this case, we find that, when the fluxes are the same in both rings, the current for $g = 2$ has the same periodicity as that of a single ring, i.e. $I^{(g=2)}(\Phi + \Phi_0) = I^{(g=2)}(\Phi)$. At temperatures $T \leqslant 0.02$ K, the lineshape of the current as a function of the flux is of a sawtooth form (see figure 1(b)). For other series $N_s \neq 2m + 2$, the periodicity of the current is complicated. Using the experimental values given in the experiment [17], we estimate that the number of sites in our model should be in the range of $N_s \approx 230$, the hopping constant should be $t = \frac{\hbar v_F}{2a \sin(K_F a)} \approx 0.01$ eV and the temperature in the experiment should be $T = 0.02$ K. Using these units, we find that the persistent current is given in terms of a dimensionless current I (see figures 1(b) and (c)) with the actual current value $I^{(g=2)} = I \times 0.92 \times 10^{-4}$ A.

4.1. Equal fluxes for $g = 2$

For this case the secular equation is simplified and takes the form of $4[\Delta^{(\mathrm{even})}(\hat{\varphi}, \lambda)\Delta^{(\mathrm{odd})}(\hat{\varphi}, \lambda)] = 1$.

For $N_s = 2$ we solve analytically the secular equation. We find that the eigenvalues are given by $\lambda(n, \varphi; N = 2) = r(N = 2)\epsilon(n, \varphi; N = 2)$ where $\epsilon(n, \varphi, N = 2) = -2t \cos[\frac{2\pi}{N=2}(n + \hat{\varphi})]$ and $n = 0, 1$ are the single-ring eigenvalues. The value for $r(N = 2)$ is $r(N = 2) = \frac{\sqrt{3}}{2}$. To find the eigenvalues for other numbers of sites $N_s = 6, 10, 14, 18, 22, 26, 30$, we numerically find the relation $\lambda(n, \varphi; N) = r(N)\epsilon(n, \varphi; N)$ where $n = 0, 1, \ldots, N - 1$ and $\epsilon(n, \varphi; N) = -2t \cos[\frac{2\pi}{N}(n + \hat{\varphi})]$ are the single-ring eigenvalues.

The function $r(N)$ is given in figure 1(a). This figure shows that the function $r(N)$ reaches *one* for large N. Using the function $r(N)$ given in figure 1(a) we compute the current for the $g = 2$ case as a function of temperature, $I^{(g=2)}(\varphi; N; T) = -\sum_{n=0}^{n=N-1} \frac{\mathrm{d}}{\mathrm{d}\varphi}[r(N)\epsilon(n, \varphi; N)] F(\frac{r(N)\epsilon(n,\varphi;N) - E_F)}{K_B T})$.

Figure 1(b) represents the current for $N_s = 30$ sites at two temperatures $T = 0.02$ and $T = 20$ K. In this figure the current is given in dimensionless units I plotted as a function of the dimensionless flux $f \equiv \hat{\varphi}_\alpha = [-0.5, 0.5]$ ($\varphi_\alpha = 2\pi\hat{\varphi}_\alpha = [-\pi, \pi]$). The solid line represents the single-ring current and the dashed line represents the current for the genus $g = 2$ case. In figure 1(b) the ratio of the currents at $T = 0.02$ K is $r(N = 30, T = 0.02) = 0.979$.

Figure 1(c) shows that the currents at $T = 20$ K are in the range of 7 nA and the reduction of the current is larger in comparison with the $T = 0.02$ K case given in figure 1(b).

4.2. Two coupled rings with opposite fluxes, i.e. $\hat{\varphi}_1 = -\hat{\varphi}_2$

For $N_s = 2$, the eigenvalues are the same as the one obtained for the same flux case. For $N_s = 6, 10, 14, \ldots, 2m + 2$ we solve the secular equation given in equation (8) and compute the eigenvalues. In figure 2(a) we plot the total energy as a function of the opposite fluxes at $T = 0.02$ K for 30 sites $E^{(g=2)}(-\hat{\varphi}, \hat{\varphi}, N_s = 30, T = 0.02, K) = \sum_{n=0}^{n=N-1} [\lambda(-\hat{\varphi}, \hat{\varphi}) F(\frac{(\lambda(-\hat{\varphi},\hat{\varphi}) - E_F)}{K_B T})]$. The total energy dependence on the opposite flux is chaotic due to the interference between paths which encircle a zero and a non-zero flux for the two rings. In addition we observe periodic oscillation with the fundamental period Φ_0 (see figures 2(a) and (b)). For comparison we show in figure 2(b) the total energy for equal fluxes which is parabolic and the current is linear (for small fluxes).

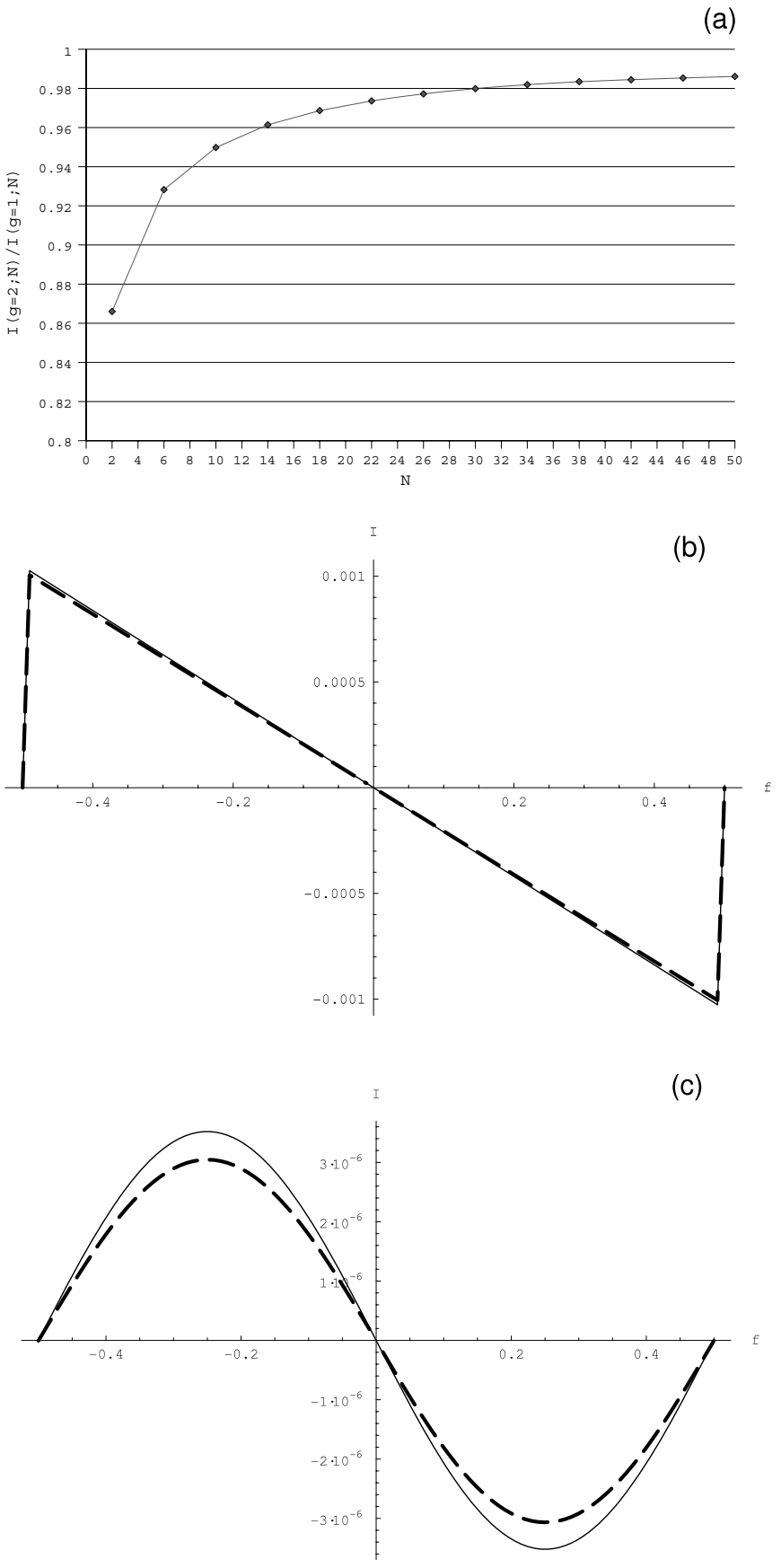

Figure 1. (a) The ratio of the double-to single-ring currents $I(g = 2; N)/I(g = 1; N) = r(N)$; (b) the single-ring (solid line) and the double-ring (dashed line) currents for $N_s = 30$ at $T = 0.02$ K; and (c) the single-ring (solid line) and the double-ring (dashed line) currents for $N_s = 30$ at $T = 20.0$ K.

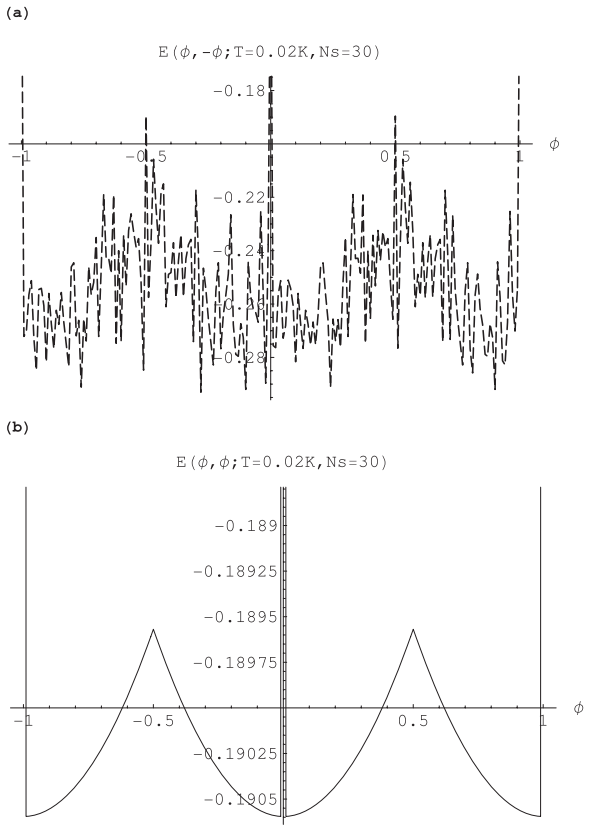

Figure 2. (a) The total energy for opposite fluxes, $\phi = \hat{\varphi}_1 = -\hat{\varphi}_2$ for 30 sites at $T = 0.02$ K $E^{(g=2)}(-\phi, \phi; N_s = 30, T = 0.02$ K); and (b) the total energy for equal fluxes, $\phi = \hat{\varphi}_1 = \hat{\varphi}_2 E^{(g=2)}(\phi, \phi; N_s = 30, T = 0.02$ K).

The effect of a finite width in the ballistic regime will give rise to a multichannel persistent current which can be analyzed using the discussions for the multichannel case which will be presented in the next section. The energy will be given by a sum of energies with respect to the different channels. Therefore we expect that the multichannel effect will give rise to a smooth function of the energy as a function of the opposite fluxes.

5. Discussions

5.1. A possible experimental confirmation of our results

At this stage it is not clear if a genus $g = 2$ experiment exists in the ballistic regime where our theory can be applied. The closest experiment which might be relevant to our theory is the experiment presented in [17]. The author of [17] make the statement (see the last sentence in their paper) that from the theoretical side a model for the ballistic regime is needed for a direct comparison with the experiment. They make this statement based on their results which indicate that the ratio between the single-ring current and the current in 16 coupled rings is close to one! The authors of [17] claim that the conditions in their experiment are not in a clear diffusive region. Therefore we believe that the ballistic theory presented can explain most of the experimental results. A full comparison with the experiment demands to estimate the effects of dephasing controlled by the dephasing time τ_ϕ and elastic scattering time τ_e. In the ballistic regime with a few impurities we can consider a model of a single $2K_F$ impurity scattering. For the opposite situation the effect of the many impurities requires us to perform an ensemble average over disorder. In this case it is not possible to work with a fixed chemical potential; instead we have to vary the chemical potential with the variation of the external flux in order to keep the number of particles fixed.

The effects of dephasing and disorder fluctuations are outside the scope of this paper. Therefore the comparison with the experimental results might be incomplete.

The experiment presented in [17] is based on a system of 16 GaAs/GaAlAs coupled rings. At this stage we have only results for two rings: we find that for two rings the ratio between the amplitudes of the currents are $I^{(g=2)}/I_{\text{single-ring}} = 0.987$. Since the experiment was performed on 16 rings we use a scaling argument in order to extrapolate the results to 16 rings. For two rings plus a scaling argument we obtain $r = I_{16-\text{rings}}/I_{\text{single-ring}} \approx [I^{(g=2)}/I_{\text{single-ring}}]^4 = [r(T = 0.02, N_s = 50)]^4 = [0.987]^4 = 0.95$. The value $r = 0.95$ is in the range of the experimental observation reported in [17]. In the experiment the rings are connected through arms of length of the order of the wavelength which can be approximated by point contact between the rings.

Next we will evaluate the amplitude of the currents. At low temperature and large number of sites the ratio of the amplitudes for the single and double ring is close to one! In order to estimate the amplitude we can consider only the amplitude for a single ring.

Using the experimental values for the Fermi velocity $v_F = 3.16 \times 10^5$ m s^{-1}, Fermi wavelength $\lambda_F = 3.5 \times 10^{-8}$ m and the ring perimeter $L = 1.2 \times 10^{-5}$ m we compute the persistent current amplitude for a single ring. Using a model of a single conducting channel we find that the amplitude current at $T = 0$ is given by $\frac{ev_F}{L} = 4.2$ nA instead of 0.4 nA reported by the experiment in [17]. Using the velocity and Fermi wavelength we obtain the effective hopping constant used in our simulations $t = \frac{\hbar v_F}{2a \sin(K_F a)}$ with $K_F = \frac{2\pi}{\lambda_F}$ and an effective lattice constant $a = \lambda_F/4$. This gives that the effective length used in the simulation for 30 sites was $L_{\text{simulation}} = 30 \cdot \lambda_F/4 = 2.6 \times 10^{-7}$ m. Our simulation shows that the current for 30 sites at $T = 0.04$ K was $I = 10^{-3}$ in dimensionless units which corresponds to a current $I \to I \cdot 0.92 \times 10^{-4}$ A. Using the value of the hopping constant $t = 1.9$ J $= 0.012$ eV we find for $N_s = 30$ a current $I(N_s = 30) = \frac{e}{\hbar}\frac{2t}{N_s} = \frac{2.8}{N_s=30} \times 10^{-6} = 92$ nA.

In figure 3 we show the current dependence on the length of the ring for single and double rings at different temperatures. At $T = 0.02$ K the current decreases linearly with length. This allows us to use at $T = 0$ a linear relation between the currents for different ring sizes. We find $\frac{I(L_{\text{simulation}})}{I(L)} = \frac{L}{L_{\text{simulation}}}$. Therefore the current obtained in the simulation $I(L_{\text{simulation}} = 2.6 \times 10^{-7}$ m$) = 92$ nA will correspond to a current for the rings with the length $L = 1.2 \times 10^{-5}$ m which is $I(L = 1.2 \times 10^{-5}$ m$) = 4.2$ nA. This means that the current of 92 nA obtained in the simulation corresponds to a current of 4.2 nA

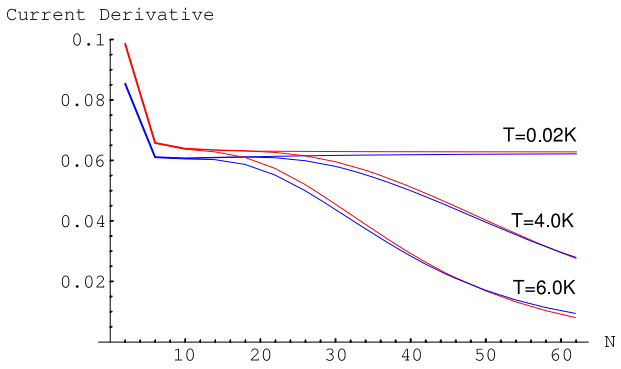

Figure 3. The current derivative with respect to the flux at $\phi \ll 1$ multiplied by the number of sites N as a function of N at different temperatures for a single and a double ring.

(This figure is in colour only in the electronic version)

in the rings. The value observed in the experiment was 0.4 nA which is a factor of 10 smaller than the prediction for a single channel and no impurities.

5.2. The effect of disorder and interactions in a multichannel model can give the correct value of the persistent current in agreement with the experiment in [17]

We explain the discrepancy of a factor of 10 between the experimental result and the theoretical calculation using a single $2K_F$ impurity model. The impurity gives rise to $2K_F$ scattering for a multichannel one-dimensional ring at zero temperature. We consider a situation of independent one-dimensional channels. This approximation is valid for a dilute impurity scattering with a large elastic scattering time τ_e and a large dephasing time τ_ϕ which obeys $\tau_\phi > \tau_e$. In addition we demand that the level spacing Δ satisfies $\Delta > \frac{\hbar}{\tau_\phi}$.

Contrary to the transport current the persistent current is without dissipation. The persistent current is determined by the time derivative of the zero mode coordinate (the macroscopic phase of the wavefunction). In [19] (Schmeltzer and Berkovits) it was shown that the effect of a $2K_F$ impurity scattering in a Luttinger liquid gives rise to an enhancement of the kinetic mass and, as a result, the persistent current is suppressed. Using the results given in figure 1 ([19] Schmeltzer and Berkovits) we find that the suppression factor for one channel was $1 - bW^2$, where b is determined by the strength of the electron–electron interactions and W is the $2K_F$ impurity scattering potential. The value of b for weak electron–electron interactions extracted from [19] was $b = 38.29$. The value of W can be obtained from the elastic transport mean free path l_e. From the conductivity measurements one finds $l_e = 8\times^{-6}$ m. Using the Boltzmann equation we will extract from the transport time the scattering potential W:

$$\frac{v_F}{l_e} = \frac{2\pi}{\hbar}\frac{t^2}{E_F}\left(\frac{W}{t}\right)^2 \langle(1-\cos(\theta))\rangle.$$

The angular average $\langle(1-\cos(\theta))\rangle$ is determined by the ratio between the width and length of the ring. We have $\langle 1-\cos(\theta)\rangle = \int_{-\theta_0}^{\theta_0}(1-\cos(\theta))\frac{d\theta}{2\pi} + \int_{\pi-\theta_0}^{\pi+\theta_0}(1-\cos(\theta))\frac{d\theta}{2\pi}$ with $\theta_0 = \frac{d}{L_{eff}}$ where $d = 10^{-6}$ m is the width of the ring (which is actually a square with the length $L_{eff} = \frac{L}{4}$). Solving this equation we find that the scattering potential is given by $\frac{W}{t} = (\frac{\lambda_F}{l_e}\frac{d}{L}\frac{\pi}{32})^{1/2} = 0.16$.

Using the result given in [19] (Schmeltzer and Berkovits) we compute the persistent current for a single ring (with one channel) in the presence of the $2K_F$ potential $\frac{W}{t} = 0.16$.

We find $\frac{ev_F}{L}(1 - bW^2) = 4.2$ nA $\cdot$ $(1 - 38.26 \cdot (0.16)^2) = 0.09$ nA which is smaller than the current observed in the experiment of 0.4 nA. This discrepancy suggest that the finite width of the rings gives rise to a multichannel persistent current. For a single ring the wavefunction is given by $\Psi(x, y) = \sum_{n=1}^{n_{max}} \psi_n(x)\chi_n(y)$ where $\chi_n(y) = (\frac{2}{d})^{1/2}\sin(n\frac{\pi}{d}y)$ is the standing wave in the transversal direction of the ring $0 \leqslant y \leqslant d$. In the absence of disorder the problem is replaced by n independent one-dimensional channels with a shifted energy $\frac{\hbar^2}{2m}(n\frac{\pi}{d})^2$ and operators $\psi_n(x)$, $\psi_n^\dagger(x)$. As a result in the absence of disorder the current will be given by

$$I = \frac{ev_F}{L}\sum_{n=1}^{n_{max}}\left(1 - \left(n\frac{\pi}{d}\frac{\lambda_F}{2\pi}\right)^2\right)^{1/2}.$$

The effect of the $2K_F$ impurity scattering enhances the mass for different channels. This enhancement is a function of the Fermi velocity for each channel. Therefore the current will be replaced by

$$I = \frac{ev_F}{L}\sum_{n=1}^{n_{max}}\left(1 - \left(n\frac{\pi}{d}\frac{\lambda_F}{2\pi}\right)^2\right)^{1/2} \times \left[1 - bW^2\left(1 - \left(n\frac{\pi}{d}\frac{\lambda_F}{2\pi}\right)^2\right)^{-1}\right].$$

We have replaced b by $b(1 - (n\frac{\pi}{d}\frac{\lambda_F}{2\pi})^2)^{-1}$ in agreement with [19] where it was shown that b is a function of the Fermi velocity for each channel. Using the extracted values of $b = 38.26$ and $W = 0.16$ we find from the condition $[1 - bW^2(1 - (n\frac{\pi}{d}\frac{\lambda_F}{2\pi})^2)^{-1}] \geqslant 0$ that the maximum number of conducting channels is given by $n_{max} = 8$. Substituting the value of $n_{max} = 8$ in the last current equation allows us to compute the current. We find that the persistent current is given by $I = 0.4$ nA which is in agreement with the experiment in [17].

5.3. The constraints for two rings with a finite width

For this case we have to do the gluing for two rings with a width d. The transversal direction is y and x is the direction for the one-dimensional channel. The constraints used previously are modified in the following way: $Q(x = 0, y = d + \epsilon) = C_1(L/2, y = d + \epsilon') - C_2(0, y = d + \epsilon'')$ and $Q^+(0, y = d + \epsilon) = C_1^+(L/2, y = d + \epsilon') - C_2^+(0, y = d + \epsilon'')$, where $\epsilon' + \epsilon'' = l_{arm}$ and $l_{arm} = d$ represents the length of the arm which connects the two rings. As a result the common part between the rings will be given by the connecting arm $l_{arm} = d$.

The second-class constraints is given by

$$[Q(x \approx 0, y = d + \epsilon'), Q^+(x' \approx 0, y = d + \epsilon'')]_+ = 2\delta(\epsilon' + \epsilon'' - l_{arm}).$$

Using the second-class constraint we obtain the equation of motion for the two-rings spinor $C_{\alpha=1,2}(x, y)$:

$$\begin{aligned} i\hbar\frac{dC_\alpha(x,y)}{dt} &= [C_\alpha(x,y), H_T]_D = [C_\alpha(x,y), H] \\ &- \int dz \int dz' [C_\alpha(x,y), Q^+(0, y = d+z]_+ \\ &\times ([Q^+(0, y = d+z), Q(0, y = d+z']_+)^{-1} \\ &\times [Q(0, y = d/2+z'), H] \\ &- \int dz \int dz' [C_\alpha(x,y), Q(0, y = d+z]_+ \\ &\times ([Q(0, y = d+z), Q^+(0, y = d+z')]_+)^{-1} \\ &\times [Q^+(0, y = d+z'), H]. \end{aligned}$$

In the absence of disorder we can simplify the expression for the constraints if we introduce a contact point $y = d_0$ chosen such that $d < d_0 < l_{\rm arm}$. As a result we obtain a constraint for each channel, $Q_n(x = 0)$ and $Q_m(x = 0)$. We have the conditions for the second-class constraints, $[Q_n, Q_m^+]_+ = 2\delta_{(n,m)}$. As a result the current for two rings will be similar to the one-dimensional case. The difference between the current for different channels in two rings will be determined by the transversal shift potential energy, $\frac{\hbar^2}{2m}(n\frac{\pi}{d})^2$. We have repeated our simulation for different transversal potentials with $n < n_{\rm max} = 8$ (the number of propagating zero modes) and find that the ratio between the double and single rings remains one.

5.4. Generalization to many coupled rings

In order be able to study a system of many coupled rings we have to find a method which can be applied to many coupled rings. It seems that it might be preferable to replace the constraint of the single-particle operator with the constraints defined in terms of the electronic densities and currents. This method can be combined with the basic technique of matching the boundary conditions used for solving quantum wire problems.

At the common point of the two rings at $x = 0$ the constraint gives rise to equal densities, $\rho_1(x = 0) = \rho_2(x = 0)$. Formally this condition is enforced with the help of the scalar field $a_0\delta(x)$ which plays the role of a Lagrange multiplier. For g coupled rings we introduce a set of scalar potentials $a_0^{(0,1)} \dots a_0^{(g-2,g-1)}$ for which a statistical annealed average has to be performed.

The Hamiltonian for g coupled rings with $\rho_n(x)$ electronic density and $\varphi_1 \dots .\varphi_g$ fluxes is:

$$\begin{aligned} H^{(g)} &= H\left(\varphi_1 \dots \varphi_g\right) + \sum_{n=0}^{g-2} a_0^{(n,n+1)} \big(\rho_n\left(x - (n+1)L\right) \\ &- \rho_{n+1}\left(x - nL\right)\big). \end{aligned}$$

This formulation allows us to use standard field theory methods such as bosonization and renormalization group [19, 20]. Using the bosonization method we replace the fermion field in each ring by a bosonic field. As a result the first term in the Hamiltonian $H(\varphi_1 \cdots \varphi_g)$ is replaced by g sound wave Hamiltonians. The effect of the scalar potentials $a_0^{(0,1)} \dots a_0^{(g-2,g-1)}$ is viewed as g impurities in a one-dimensional wire which are described by g sine-Gordon terms which models the contact between two connected rings. Using such a formulation the persistent current is computed.

5.5. The capacitor model $E_{\rm C} = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$ for coupled rings—a wavefunction amplitude calculation based on matching the boundary conditions

It will be interesting to see if an alternative solution not based on constraints can be found for the wavefunction. Such a formulation will allow the use of the matching boundary condition method. We will demonstrate that such a modeling leads to a non-universal description based on explicit microscopic parameters. On the other hand we will show that the physical constraints on the two rings gives rise to an universal formulation of the problem. This suggest that a direct solution based on just matching the boundary conditions is not possible.

For simplicity we will consider a continuum formulation for the two rings. We propose the following physical model. The contact between the two rings is described by a capacitor C_0. At the contact point the capacitive energy of the system will be $E_{\rm C} = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$ where C_0 is the capacitor of the dot and $\rho_1(x = 0) - \rho_2(x = 0)$ is the charge difference between the rings. The Hamiltonian for the two rings in the presence of the capacitor at the contact point is given by

$$\begin{aligned} H_{\rm C} &= H + \frac{1}{2C_0}[C_1^+(x=0)C_1(x=0) \\ &- C_2^+(x=0)C_2(x=0)]^2. \end{aligned}$$

The wavefunction for this problem can be computed using the method of matching boundary conditions. Due to the fact that the energy depends on the explicit value of the capacitance C_0 the solution for the wavefunction will not be universal.

In order to clarify this point we will describe the point contact as a constraint problem. At the point contact the electronic density operator $\rho_1(x = 0)$ on ring one is equal to the density operator $\rho_2(x = 0)$ on the second ring. Instead of using the Dirac method we can enforce the constraint by using Lagrange multipliers. The constraint is enforced by a Lagrange multiplier a_0 which act as a point contact potential for the two rings. As a result we obtain a problem of two rings with a delta function potential at the contact point $x = 0$ (we have used a symmetric description for the two rings). The Hamiltonian with the Lagrange multipliers is given by

$$\begin{aligned} H_0 &= H + a_0[C_1^+(x=0)C_1(x=0) \\ &-(C_2^+(x=0)C_2(x=0))]. \end{aligned}$$

The equivalence between the two formulations, the exact one given by the Lagrange field a_0 and the capacitor Hamiltonian $E_{\rm C} = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$, is achieved only in the limit $C_0 \to 0$. Alternatively, using the Hubbard–Stratonovich decoupling for $E_{\rm C} = \frac{(\rho_1(x=0)-\rho_2(x=0))^2}{2C_0}$ allows us to introduce the field a_0 which acts as a potential for the two rings $a_0[C_1^+(x = 0)C_1(x = 0) - (C_2^+(x = 0)C_2(x = 0))]$ with the value of the potential controlled by the capacitor. This shows

that qualitatively H_0 and H_C are similar. The major difference between the two is that in the exact constraint formulation the physical result does not depend on a particular value of the capacitor C_0. Instead we have to average over all the potentials a_0. Therefore the capacitor model is incomplete if the limit $C_0 \to 0$ is not taken.

Due to the difficulty in taking the strong coupling limit $\frac{1}{C_0} \to \infty$ we will work with the Lagrange multiplier formulation. We will investigate the Hamiltonian H_0 in the presence of the potential a_0 and compute the wavefunction. The wavefunction is given by $|\Psi_s(t)\rangle$. The physical evolution operator $U_{\rm phys}(t)$ allows us to compute the wavefunction at a time t, $|\Psi_s(t)\rangle = U_{\rm phys}(t)|\Psi_s(t=0)\rangle$. The physical operator $U_{\rm phys}(t) = e^{\frac{-i}{\hbar}H_0 t} \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \frac{da_0(x=0)}{2\Lambda} \exp(\frac{-i}{\hbar} a_0(x = 0)\rho(x = 0))$ is controlled by the constraint which is implemented by a time-independent integration of the Lagrange multiplier $a_0(x = 0)$.

This shows that the physical result is given by the evolution operator $U_{\rm phys}(t)$. This means that we have to compute the current for all possible values of the point contact potential and in addition we have to perform an annealed average over all the point contact potentials.

The advantage of the $U_{\rm phys}(t)$ operator is that it maps the problem of coupled ring into an impurity potential (with varying strength) for which the method of matching boundary conditions can be used.

Explicitly the solution of the problem in a symmetric configuration with the common point at $x = 0$ is solved in the following way. We fold the space of the first ring from $[-L, 0]$ to $[L, 0]$ such that the space of the two rings is restricted to $0 \leqslant x \leqslant L$. The wavefunction for the coupled rings $Z_E(x)$ with eigenvalue E is given as a spinor with two components $Z_\alpha(x)$, where $\alpha = 1, 2$ is the index for the two rings. The Schrödinger equation for the two rings in the presence of the point contact potential $a_0\delta(x)$ is

$$\left[-\left(-\partial_x - i\frac{2\pi}{L}\varphi_1\right)^2 + a_0\delta(x)\right] Z_1(x) = K^2 Z_1(x)$$

$$\left[-\left(\partial_x - i\frac{2\pi}{L}\varphi_2\right)^2 - a_0\delta(x)\right] Z_2(x) = K^2 Z_2(x)$$

where $K^2 \equiv \frac{2m}{\hbar^2}E$ with the energy E. Using these equations we can compute the current for the two rings in the presence of the point contact potential. The current is a function of the strength of the point contact potential a_0 and the two fluxes φ_1, φ_2 given by $I[\varphi_1, \varphi_2; a_0]$. The physical current is obtained after averaging over all possible values of the point contact potentials a_0, $\bar{I}(\varphi_1, \varphi_2) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \frac{da_0}{2\Lambda} I(\varphi_1, \varphi_2; a_0)$.

The eigenfunctions for this problem are obtained by matching the boundary conditions:

(a) The continuity condition:

$$Z_1(x) = Z_1(x + L) \qquad \text{and}$$
$$Z_2(x) = Z_2(x + L)$$

(b) The discontinuity of the derivative at $x = 0$ for each ring:

$$\left(-\partial_x - i\frac{2\pi}{L}\varphi_1\right) Z_1(x = -\varepsilon)$$
$$- \left(-\partial_x - i\frac{2\pi}{L}\varphi_1\right) Z_1(x = \varepsilon) = a_0 Z_1(x = 0)$$
$$\left(\partial_x - i\frac{2\pi}{L}\varphi_2\right) Z_2(x = -\varepsilon)$$
$$- \left(-\partial_x - i\frac{2\pi}{L}\varphi_2\right) Z_2(x = \varepsilon) = -a_0 Z_2(x = 0).$$

(The change of sign of the flux in ring one $\alpha = 1$ is a result of folding the space from $[-L, 0]$ to $[L, 0]$.) The solution for the wavefunction $Z_E(x)$ for equal fluxes is given by

$$Z_E(x) = \frac{1}{\sqrt{2\left(1 + |R|^2\right)}} \left\{\delta_{\alpha,1}\left[e^{-iKx} - e^{iKx}R(-K, \varphi)\right] + \delta_{\alpha,2}\left[e^{-iKx} - e^{iKx}R^*(-K, \varphi)\right]\right\}.$$

From the two boundary conditions we determine the function $R(K, \varphi) = \frac{1-e^{-i(KL+2\pi\varphi)}}{1-e^{i(KL-2\pi\varphi)}}$. This function depends on the flux φ and the eigenvalue K.The eigenvalue K is $K = \frac{2\pi}{L}n + \frac{2\pi}{L}\delta(\frac{a_0}{2K}, \varphi)$ where $n = 0, \pm 1 \ldots$ and the phase shift $\delta(\frac{a_0}{2K}, \varphi)$. The solution of the phase shift is determined by the flux φ and the ratio $\frac{a_0}{2K}$.

The explicit solution of the phase shift δ is given by the following trigonometric equation:

$$\cos\left[2\pi\delta - \arctan\left(\frac{a_0}{2K}\right)\right] = \cos[\varphi]\cos\left[\arctan\left(\frac{a_0}{2K}\right)\right].$$

The value of the phase shift allows us to compute the current $I[\varphi; a_0]$ for a fixed value of the potential a_0. Therefore this will not be a universal result.

The physical current will be obtained after the averaging over all the potentials a_0 is performed:

$$\bar{I}(\varphi) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \frac{da_0}{2\Lambda} I(\varphi; a_0).$$

The need for the additional average makes the solution for the persistent current more involved and practically difficult. This shows that no simple method based only on matching the boundary condition can be found! Comparing Dirac's method used in the first part of this paper with the formulation in this section we conclude that Dirac's method is more advantageous since it does not require additional averages.

6. Summary

In this paper we have introduced a method which solves the quantum mechanical problem for high genus materials. The global phase of the wavefunction for geometrical structures such as double rings has been computed using the method of constraints. This method is applicable to a variety of problems where coherence of the wavefunction is important. As an explicit example we have presented the first solution for persistent currents in two coupled rings. Using numerical simulations we have computed the current dependence on the flux, temperature and the number of sites. This theory might be tested in coupled rings for equal and opposite fluxes in the ballistic regime. A possible explanation of the experiment in [17] based on the ballistic theory has been proposed.

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